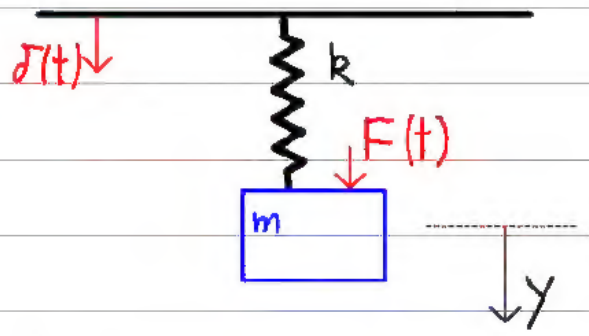
Meriam 8/3 - 8/4: Forced undamped vibrations

$\delta(t) = \delta_0 \sin(\omega_0 t)$

or $F(t) = F_0 \sin(\omega_0 t)$

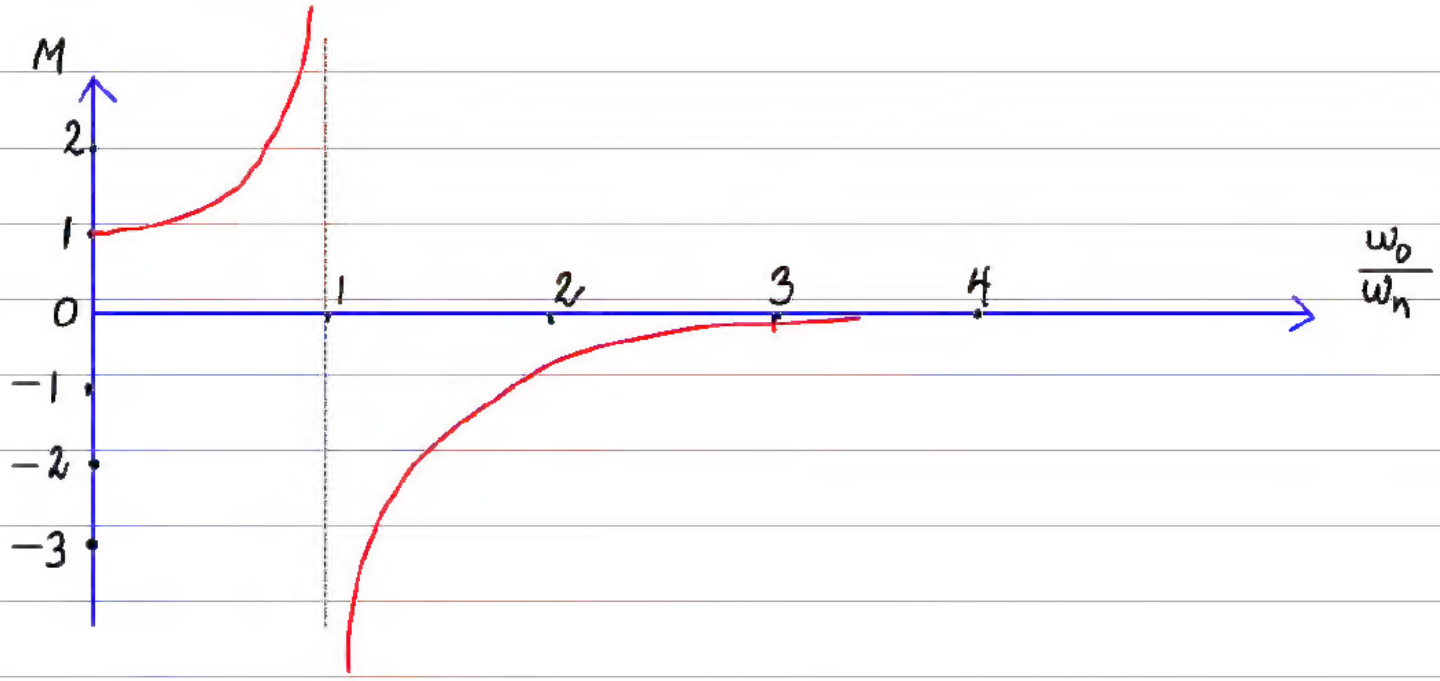


$$\ddot{y} + \frac{k}{m} y = \frac{k\,\delta(t)}{m}$$

$$y(t) = \left(\frac{1}{1 - (\omega_0/\omega_n)^2}\right) \delta_0 \sin\omega_0 t$$

F_0/k

M — static deflection

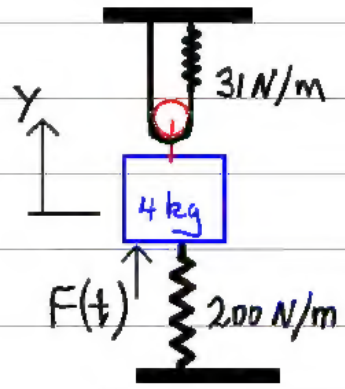


critical (resonant) driving frequency ω_0 is ω_n

Eg.

$F(t) = F_0 \sin(\omega_0 t)$,

where $F_0 = 15\ N$



Find oscillation amplitude for

(a) $\omega_0 = 6\ s^{-1}$

(b) $\omega_0 = 12\ s^{-1}$

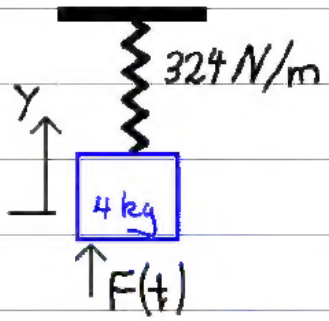
Solution

(i) Define upwards direction as +ve y-axis. Set origin where gravity and spring forces add to 0.

(ii) Combine springs: $k_{total} = 200 + (4 \times 31) = 324\ N/m$

(iii) $\sum F_y = -k_{total}\, y + F_0 \sin(\omega_0 t) = m\ddot{y}$

$$\rightarrow \quad \ddot{y} + \frac{324\ N/m}{4kg}\, y = \frac{15\ N}{4\ kg} \sin \omega_0 t$$



(iv) $\omega_n^2 = \frac{324}{4} = 81\ s^{-2}$; $\omega_n = 9\ s^{-1}$

(v) steady state solution:

$$y(t) = \text{static deflection} \times M \times \sin(\omega_0 t)$$

$$y(t) = \frac{15\ N}{324\ N/m} \times \frac{1}{1 - \omega_0^2/\omega_n^2} \times \sin(\omega_0 t)$$

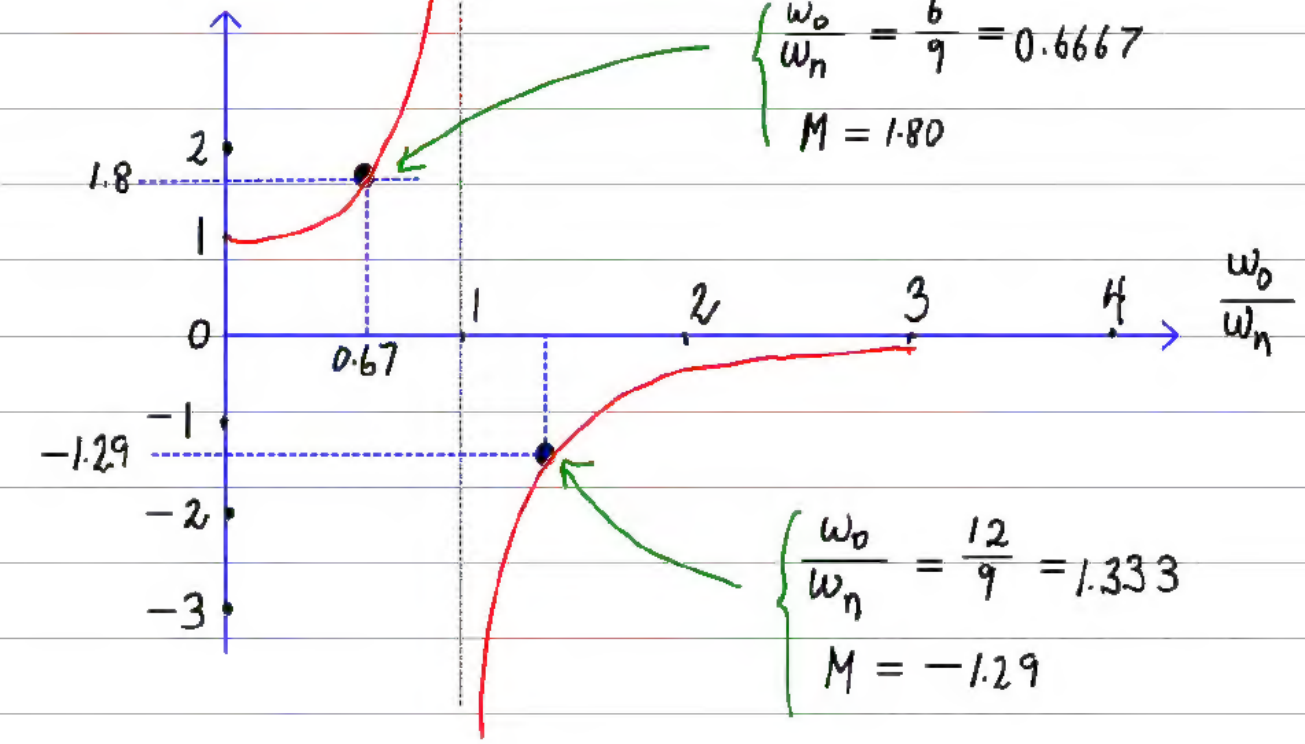
(vi) Amplitude $= \left| \left(\frac{15}{324}\right) \left(\frac{1}{1 - \omega_0^2/81} \right) \right|$

0.046 meters (static def); M

For $\omega_0 = 6\ s^{-1}$, Amplitude $= |0.046)\ \underset{M}{(1.80)}| =$ **0.083 meters**

For $\omega_0 = 12\ s^{-1}$, Amplitude $= |(0.046)\underset{M}{(-1.29)}| =$ **0.06 meters**

$M = \frac{1}{1 - \omega_0^2/\omega_n^2}$



E.g.

Given: $k = 1000\ N/m$

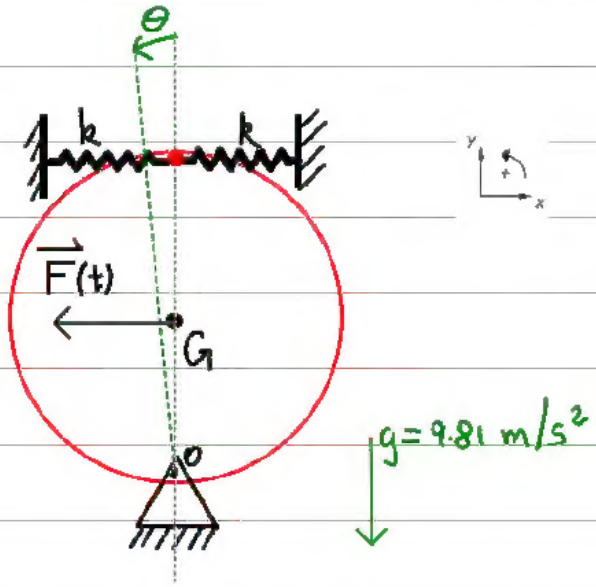
$m = 50\ kg$

$R = 2\ m$

$k_G = 1.5\ m$

$\vec{F}(t) = 50 \sin(8t)$ Newtons $(-\hat{i})$

Diagram shows neutral position



Find:

- Equation of motion for $\theta(t)$ (steady state)
- Amplification Factor M
- ω_n
- θ_{max} = amplitude of vibration

Solution:

$$\theta(t) = \underbrace{\theta_s}_{\text{static deflection}} * \underbrace{\frac{1}{(1 - \omega_0^2/\omega_n^2)}}_{M} * \sin(\omega_0 t) + \dots$$

(ω_0 = 8)

→ Need to find:
- ω_n → M
- static deflection θ_s → θ_{max}

Given: $k = 1000\ N/m$

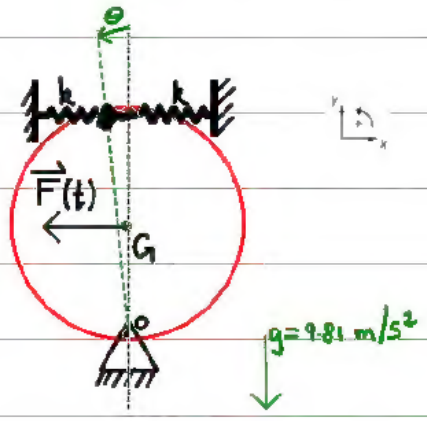
$m = 50\ kg$

$R = 2\ m$

$k_G = 1.5\ m$

$\vec{F}(t) = 50\sin(8t)$ Newtons $(-\hat{\imath})$

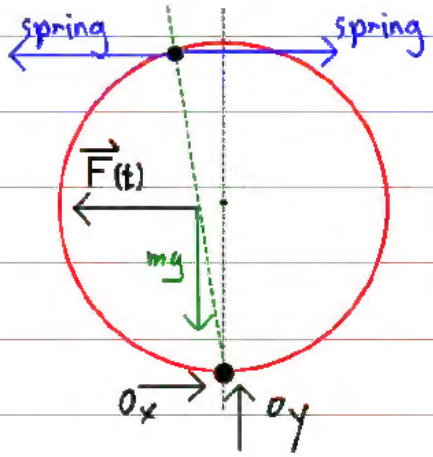
Diagram shows neutral position



Rotate disk a small angle θ: springs are deformed by $2R\theta$

→ change in spring force is $2kR\theta$

→ change in spring moment$_o$ is $-(2kR\theta)(2R)$ x 2 springs



$$\sum M_o = -8kR^2\theta + mgR\theta + 50\sin(8t)R = I_o\ddot{\theta}$$

$$\ddot{\theta} + \frac{8kR^2 - mgR}{I_o}\theta = \frac{50R}{I_o}\sin 8t$$

$$\rightarrow \omega_n^2 = \frac{(8kR^2 - mgR)}{mk_G^2 + mR^2} = 99.26$$

$$\omega_n = 9.96\ s^{-1}$$

$$M = \frac{1}{1 - \omega_0^2/\omega_n^2} = 2.82$$

($\omega_0^2 = 8^2$, $\omega_n^2 = 9.96^2$)

Static deflection θ_s: $\sum M_o = 0$

$$(50)(R) - 8kR^2\theta_s + mgR\theta_s = 0$$

$$\rightarrow \theta_s = 0.0032 \text{ radians}$$

$$\rightarrow \theta(t) = (0.0032)(2.82)\sin(8t) \quad (+\ c\sin(9.96t + \phi))$$

(θ_s, M)

What happens if we add a damper to forced vibrations?

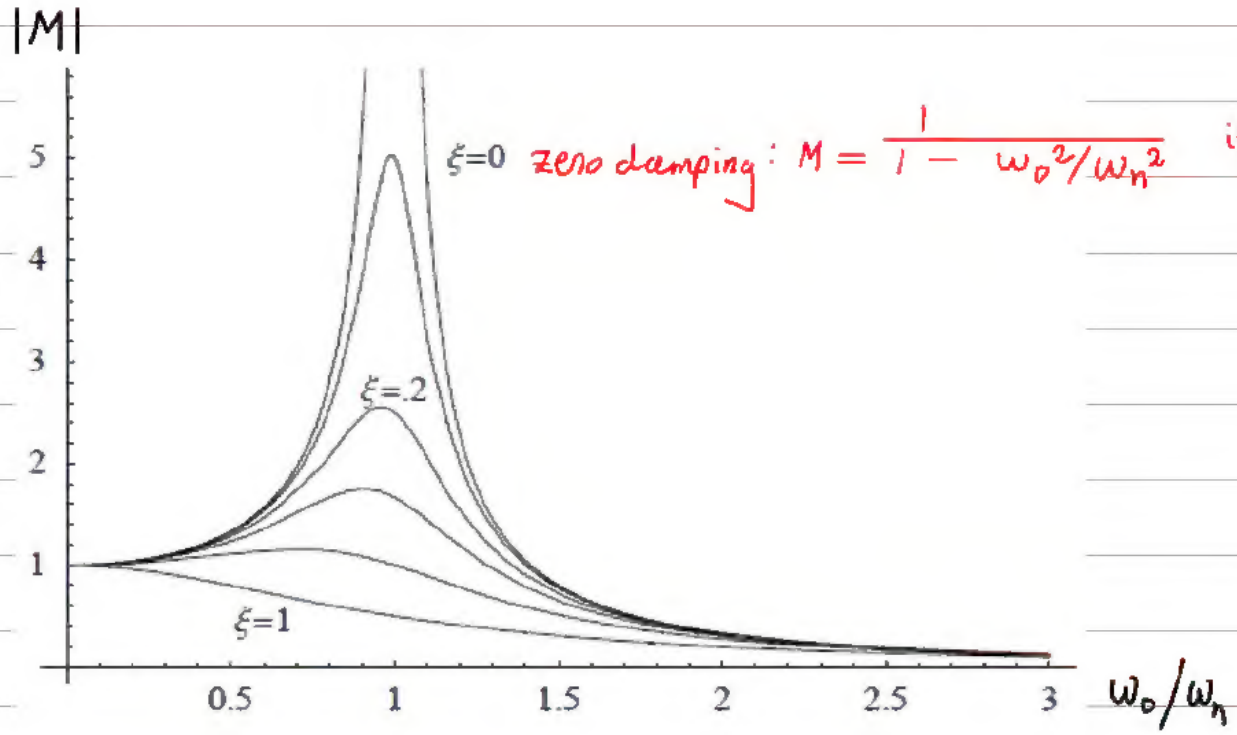
$$m\ddot{x} + c\dot{x} + kx = F_0 \sin\omega_0 t$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = (F_0/m)\sin\omega_0 t$$

$$\rightarrow \quad \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = (F_0/m)\sin\omega_0 t$$

For small ζ,

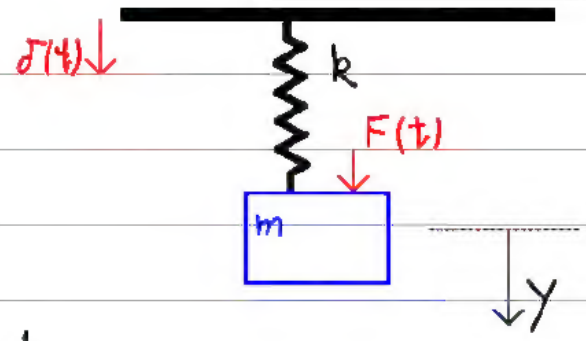
$$x(t) \approx (\text{static deflection})\,(M)\,(\sin\omega_0 t) + \cancel{(A\sin(\omega_n t + \phi))}$$



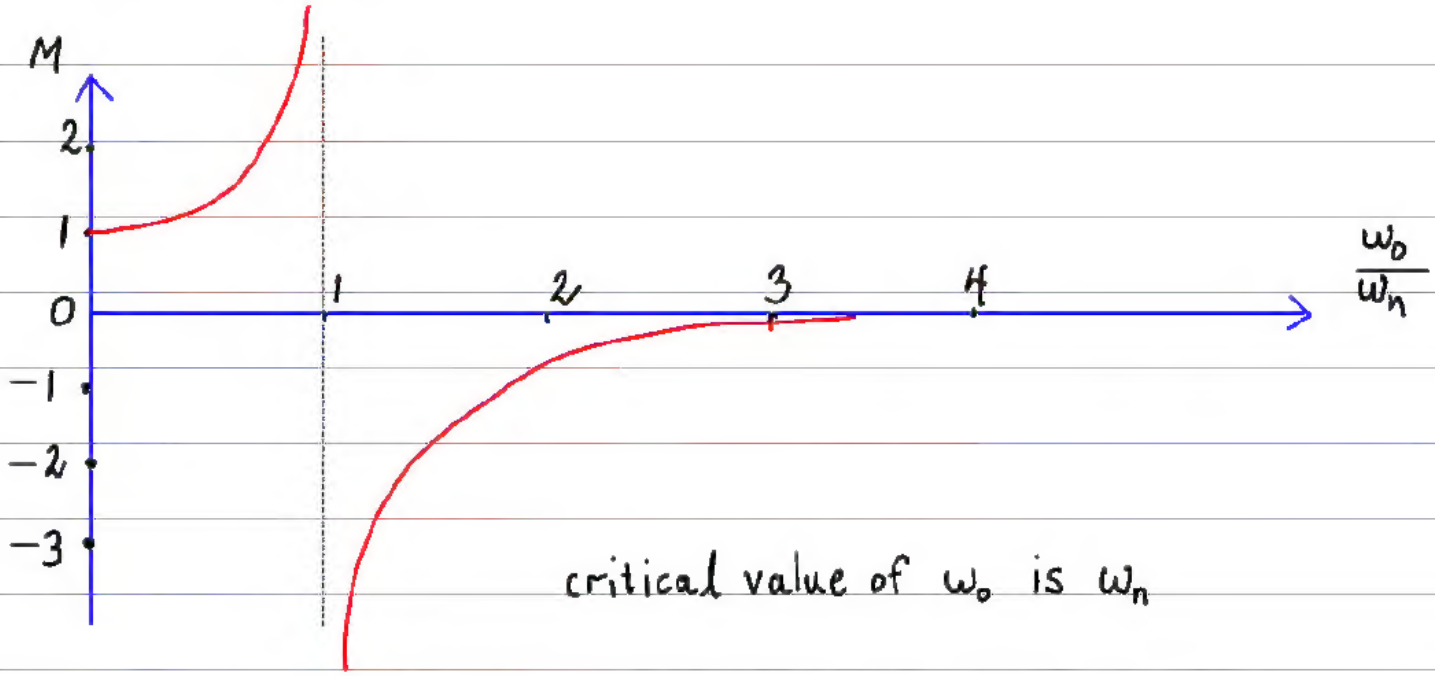
$\xi=0$ zero damping: $M = \dfrac{1}{1 - \omega_0^2/\omega_n^2}$ if $\zeta = 0$

Summary: Forced undamped vibrations

- Forcing function: $\delta(t) = \delta_0 \sin(\omega_0 t)$ or $F(t) = F_0 \sin(\omega_0 t)$



- $\ddot{y} + \frac{k}{m} y = \frac{k\delta(t)}{m}$
- $y(t) = \left(\frac{1}{1 - (\omega_0/\omega_n)^2}\right) \delta_0 \sin\omega_0 t$

F_0/k

M — static deflection